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TWOFOLD UNBRANCHED COVERINGS OF GENUS TWO 3-MANIFOLDS ARE HYPERELLIPTIC

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ABSTRACT

A closed 3-dimensional manifold is hyperelliptic if it admits an involution such that the quotient space of the manifold by the action of the involution is homeomorphic to the 3-sphere. We prove that a twofold unbranched covering of a genus two 3-manifold is hyperelliptic. This result is reminiscent of a theorem, which seems to have first appeared in a paper by Enriques and which has been reproved more recently by Farkas and Accola, which states that a twofold unbranched covering of a Riemann surface of genus two is hyperelliptic.

1. Introduction

A closed Riemann surface is said to be **hyperelliptic** if it admits a conformal involution such that the quotient space of the surface by the action of the involution is the 2-sphere S^2. Many properties of hyperelliptic Riemann surfaces are well studied in complex analysis. For example, it is known that any Riemann surface of genus two is hyperelliptic and it admits a unique hyperelliptic involution. Another interesting result, which seems to have first appeared in a paper

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by Enriques [5] and which has been reproved more recently by Farkas [6] and Accola [1], is the following:

A twofold unbranched covering of a Riemann surface of genus two is hyperelliptic.

It should be said that most of twofold unbranched coverings of a Riemann surface of genus greater than two are not hyperelliptic ([3], [7]; related papers are also [10], [9]).

If we turn to closed 3-dimensional manifold, it is convenient to adopt a topological definition of hyperelliptic involution: more precisely, we say that a closed 3-dimensional manifold is **hyperelliptic** if it admits an involution such that the quotient space of the manifold by the action of the involution is homeomorphic to the 3-sphere S^3; in this case the involution is said to be a **hyperelliptic** involution.

In [11] it has been proved that in any of the eight three-dimensional geometries there exists a 3-manifold with a geometric hyperelliptic involution. If M is a closed hyperbolic 3-manifold, by Mostow's Rigidity Theorem, a hyperelliptic involution is equivalent to an isometric hyperelliptic involution. Even in the hyperbolic case a hyperelliptic involution is, in general, not unique (see [12], where an explicit bound on the number of hyperelliptic involutions of a hyperbolic 3-manifold is computed).

The class of 3-manifolds which naturally correspond to genus two Riemann surfaces is the class of genus two 3-manifolds. We recall that a genus n **Heegaard splitting** of a closed orientable 3-manifold N is a decomposition of N into a union $V_1 \cup V_2$ of two handlebodies of genus n intersecting in their common boundary (Heegaard surface of the splitting). The **genus of** N is the lowest genus for which N admits a Heegaard splitting.

In this note we prove the following Theorem, which should be compared with the result quoted above:

THEOREM: *A twofold unbranched covering of a genus two 3-manifold is hyperelliptic.*

2. Proof of the Theorem

Before starting the proof we introduce here some terminology about bridge-presentations and abelian branched coverings of a link in S^3.

BRIDGE-PRESENTATIONS OF A LINK. An m-**bridge presentation** of a link L in S^3 is a decomposition of the pair (S^3, L) into a union $(B_1, a_1) \cup (B_2, a_2)$, where B_i for $i = 1, 2$ is a 3-ball and a_i is a set of m arcs which is trivial in B_i. We say that L is a 3-bridge link if $m = 3$ is the minimal number m for which L admits an m-bridge presentation.

ABELIAN BRANCHED COVERINGS OF A LINK. Let L be an oriented link in S^3 with n components K_i for $1 \leq i \leq n$. Let $\pi_1(S^3 - L)$ be the fundamental group of the link complement. The link complement $S^3 - L$ admits a universal abelian covering corresponding to the Hurewicz homomorphism $\gamma: \pi_1(S^3 - L) \to H_1(S^3 - L)$.

We note that $H_1(S^3 - L)$ is a free abelian group of rank n and it is generated by the homology classes of n fixed oriented meridians $m_1, \ldots, m_n$ of $K_1, \ldots, K_n$. For any abelian group A and epimorphism $\beta: H_1(S^3 - L) \to A$ we can construct the abelian covering $M(L)$ of $S^3 - L$ corresponding to the epimorphism $\beta \circ \gamma: \pi_1(S^3 - L) \to A$ and the abelian branched covering space M of S^3 obtained by completion of $M(L)$: by an abuse of language we shall shortly say that M is an A-**branched covering** of L.

We are mainly interested in the case that $A \cong \mathbb{Z}_2^r$ is an elementary abelian 2-group of finite rank r and ψ maps the n homology classes of the meridians $m_1, \ldots, m_n$ to nontrivial elements of A. Note that if the rank r of A is equal to the number n of components of L, then the corresponding A-covering is uniquely determined.

Proof of the Theorem: A genus two closed orientable surface admits a hyperelliptic involution I which has the property that for any homeomorphism f there is a homeomorphism f' isotopic to f such that $If = f'I$ ([2], 4.4; [14]). So for any genus two Heegaard splitting of a genus two 3-manifold N there exists an orientation-preserving involution of N, which we shall call **strongly hyperelliptic**, which leaves invariant the Heegaard splitting and induces the hyperelliptic involution on the Heegaard surface. The quotient of N by this involution is topologically S^3 and its branching locus is a link, say L. The Heegaard splitting of N naturally induces a 3-bridge presentation $(B_1, a_1) \cup (B_2, a_2)$ of L, where (B_i, a_i) for $i = 1, 2$ is the quotient of the handlebody V_i. Therefore L has one, two or three components. The number of components of L is equal to $1 + \operatorname{rank}_{\mathbb{Z}_2} H_1(N, \mathbb{Z}_2)$ (see, for example, [13], Sublemma 15.4; or [4], page 135, E 9.5); thus the rank of $H_1(N, \mathbb{Z}_2)$ is at most two.

If $\operatorname{rank}_{\mathbb{Z}_2} H_1(N, \mathbb{Z}_2) = 0$ (this case actually occurs for genus two 3-manifolds) then N does not admit any twofold unbranched covering, since twofold unbranched coverings of N correspond to epimorphisms of $H_1(N, \mathbb{Z}_2)$ onto $\mathbb{Z}_2$. So we can assume that $H_1(N, \mathbb{Z}_2)$ is not trivial and we need to consider only two cases:

(i) $\mathrm{rank}_{\mathbb{Z}_2} H_1(N, \mathbb{Z}_2) = 1$ and N is the $\mathbb{Z}_2$-branched covering of a three-bridge link L with two components;

(ii) $\mathrm{rank}_{\mathbb{Z}_2} H_1(N, \mathbb{Z}_2) = 2$ and N is the $\mathbb{Z}_2$-branched covering of a three-bridge link L with three components.

(i) *L has two components*

Let K_1 and K_2 be the two components of L with meridians m_1 and m_2. Let M be the $\mathbb{Z}_2 \times \mathbb{Z}_2$-covering of L. By considering the action of the covering group $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ on M we construct a diagram of abelian coverings.

The covering group A acts on M with branch set equal to the union of the preimages $\tilde{K}_1$ and $\tilde{K}_2$ of K_1 and K_2 in M. Our first step is to find the fixed point sets of the three involutions of A.

By construction, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-covering M of L corresponds to the epimorphism ψ: $H_1(S^3 - L) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $\psi(m_1) = a$, $\psi(m_2) = b$, where a and b are two fixed generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$. So the involution, say u_1, of A corresponding to a fixes pointwise the preimage $\tilde{K}_1$ of K_1; the involution, say u_2, of A corresponding to b fixes pointwise the preimage $\tilde{K}_2$ of K_2. We finally prove that the product (u_1u_2) of the two involutions acts freely on M. Each point x of K_1 has two distinct preimages $\tilde{x}$ and $\tilde{y}$ in $\tilde{K}_1$. The group A acts on the pair $(\tilde{x}, \tilde{y})$: the involution u_1 fixes pointwise both $\tilde{x}$ and $\tilde{y}$; the two other involutions u_2 and (u_1u_2) interchange $\tilde{x}$ and $\tilde{y}$. So neither u_2 nor (u_1u_2) fixes any point of $\tilde{K}_1$. An analogous argument holds for the preimage $\tilde{K}_2$ of K_2: the involution u_2 fixes pointwise $\tilde{K}_2$; the two other involutions u_1 and (u_1u_2) do not fix any point of $\tilde{K}_2$. We conclude that the involution (u_1u_2) does not fix any point of the branch set of $\tilde{K}_1 \cup \tilde{K}_2$ of A, so it acts freely on M.

The quotient N_1 of M by the action of u_2 is the $\mathbb{Z}_2$-branched covering of L along K_1 and it corresponds to the map ϕ_1: $H_1(S^3 - L) \to \mathbb{Z}_2$ such that $\phi_1(m_1) = a$ and $\phi_1(m_2) = 0$, where a is a generator of $\mathbb{Z}_2$. The quotient N_2 of M by the action of u_1 is the $\mathbb{Z}_2$-branched covering of L along K_2 and it corresponds to the map ϕ_2: $H_1(S^3 - L) \to \mathbb{Z}_2$ such that $\phi_2(m_1) = 0$ and $\phi_2(m_2) = a$, where a is a generator of $\mathbb{Z}_2$. Finally, the quotient of M by the action of (u_1u_2) is the $\mathbb{Z}_2$-branched covering N of L along K_1 and K_2 and it corresponds to the map ϕ: $H_1(S^3 - L) \to \mathbb{Z}_2$ such that $\phi(m_1) = \phi(m_2) = a$.

The diagram of coverings is represented in Figure 1. Note that in this case, since $\mathrm{rank}_{\mathbb{Z}_2} H_1(N, \mathbb{Z}_2) = 1$, the 3-manifold N has a unique unbranched $\mathbb{Z}_2$-

covering. This covering is M and it corresponds to the unique epimorphism of $H_1(N,\mathbb{Z}_2)$ onto $\mathbb{Z}_2$.

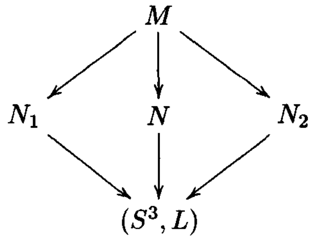

Figure 1

Since $L = K_1 \cup K_2$ admits a 3-bridge presentation, one component of L, say K_1, admits a 1-bridge presentation and the second component, K_2, admits a 2-bridge presentation. But a knot with a 1-bridge presentation is trivial. So K_1 is trivial and the $\mathbb{Z}_2$-branched covering N_1 of L along K_1 is S^3. The thesis follows from the fact that N_1 is the quotient of M by the action of u_2, so the twofold unbranched covering M of N is hyperelliptic.

(ii) *L has three components*

Let K_1, K_2 and K_3 be the three components of L with meridians m_1, m_2 and m_3. We construct a diagram of abelian coverings.

Consider first the $\mathbb{Z}_2 \times \mathbb{Z}_2$-branched covering R_{12} of L corresponding to the map $\psi_{12}\colon H_1(S^3-L) \to \mathbb{Z}_2\times\mathbb{Z}_2$ such that $\psi_{12}(m_1) = \psi_{12}(m_2) = a$, $\psi_{12}(m_3) = b$, where a and b are two fixed generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The covering group $A_{12} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on R_{12} with branch set the union of the preimages $\tilde{K}_1$, $\tilde{K}_2$ and $\tilde{K}_3$ of K_1, K_2 and K_3.

By an analysis analogous to the case of a link with two components we find the fixed point sets of the involutions of A_{12}. The fixed point set of one involution of A_{12}, say v_{12}, is $\tilde{K}_3$; the fixed point set of a second involution u_{12} is $\tilde{K}_1 \cup \tilde{K}_2$; finally, their product $(u_{12}v_{12})$ acts freely on R_{12}. The quotient N_3 of R_{12} by the action of u_{12} is the $\mathbb{Z}_2$-branched covering of L along K_3 and it corresponds to the map $\phi_3\colon H_1(S^3 - L) \to \mathbb{Z}_2$ such that $\phi_3(m_1) = \phi_3(m_2) = 0$ and $\phi_3(m_3) = c$, where c is a generator of $\mathbb{Z}_2$. The quotient N of R_{12} by the action of $(u_{12}v_{12})$ is the $\mathbb{Z}_2$-branched covering N of L along K_1, K_2 and K_3 and it corresponds to the map $\psi\colon H_1(S^3 - L) \to \mathbb{Z}_2$ such that $\psi(m_1) = \psi(m_2) = \psi(m_3) = d$, where d is a generator of $\mathbb{Z}_2$.

In an analogous way we can define the $\mathbb{Z}_2\times\mathbb{Z}_2$-branched coverings R_{13}, respectively R_{23}, of L corresponding to the maps $\psi_{13}\colon H_1(S^3-L) \to \mathbb{Z}_2\times\mathbb{Z}_2$ such that $\psi_{13}(m_1) = \psi_{13}(m_3) = a$, $\psi_{13}(m_2) = b$, respectively, $\psi_{23}\colon H_1(S^3 - L) \to \mathbb{Z}_2 \times \mathbb{Z}_2$

such that $\psi_{23}(m_2) = \psi_{23}(m_3) = a$, $\psi_{23}(m_1) = b$, where a and b are two fixed generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

The covering group $A_{13} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on R_{13} with branch set the union of the preimages $\tilde{K}_1$, $\tilde{K}_2$ and $\tilde{K}_3$ of K_1, K_2 and K_3 in R_{13}. The fixed point set of one involution of A_{13}, say v_{13}, is $\tilde{K}_2$; the fixed point set of a second involution u_{13} of A_{13} is $\tilde{K}_1 \cup \tilde{K}_3$; hence their product $(u_{13}v_{13})$ acts freely. The quotient N_2 of R_{13} by the action of u_{13} is the $\mathbb{Z}_2$-branched covering of L along K_2; the quotient of R_{13} by the action of $(u_{13}v_{13})$ is N, which is the $\mathbb{Z}_2$-branched covering of L along K_1, K_2 and K_3.

Finally, the covering group $A_{23} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on R_{23} with branch set the union of the preimages $\tilde{K}_1$, $\tilde{K}_2$ and $\tilde{K}_3$ of K_1, K_2 and K_3 in R_{23}. The fixed point set of one involution of A_{23}, say v_{23}, is $\tilde{K}_1$; the fixed point set of a second involution u_{23} of A_{23} is $\tilde{K}_2 \cup \tilde{K}_3$; hence their product $(u_{23}v_{23})$ acts freely. The quotient N_1 of R_{23} by the action of u_{23} is the $\mathbb{Z}_2$-branched covering of L along K_1; the quotient of R_{23} by the action of the involution $(u_{23}v_{23})$ is again N.

To complete this diagram of branched coverings it should be said (but we do not need it in the following) that the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-branched covering M of L is a $\mathbb{Z}_2$-covering of R_{12}, R_{13} and R_{23} and a $\mathbb{Z}_2 \times \mathbb{Z}_2$-covering of N, N_1, N_2 and N_3.

The diagram of coverings in the case that the link L has three components is represented in Figure 2. Note that in this case, since $\text{rank}_{\mathbb{Z}_2} H_1(N, \mathbb{Z}_2) = 2$, the 3-manifold N has three twofold unbranched coverings arising from different epimorphisms of $H_1(N, \mathbb{Z}_2)$ onto $\mathbb{Z}_2$. These coverings are exactly R_{12}, R_{13} and R_{23}.

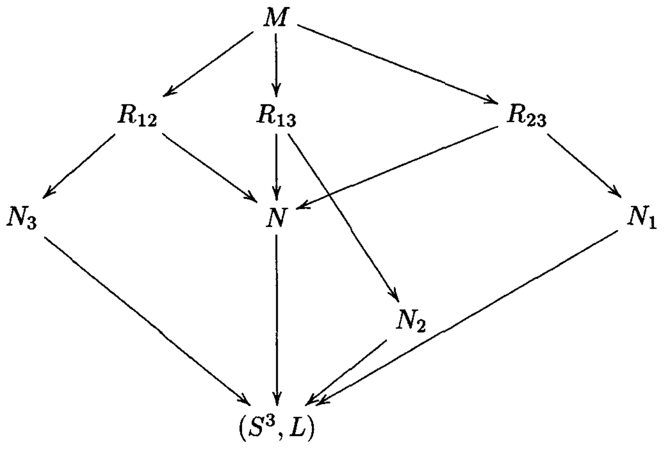

Figure 2

Since $L = K_1 \cup K_2 \cup K_3$ admits a 3-bridge presentation, each of its components

K_1, K_2 and K_3 admits a 1-bridge presentation, hence it is trivial. So the $\mathbb{Z}_2$-branched covering N_i of L along K_i for $i = 1, 2, 3$ is S^3. The thesis follows from the fact that N_i is the quotient of R_{jk} by the action of an involution, so any twofold unbranched covering R_{jk} of N is hyperelliptic.

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